United Kingdom
Mathematics Trust

Senior Mathematical Challenge

Tuesday 4 October 2022

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For reasons of space, these solutions are necessarily brief.

There are more in-depth, extended solutions available on the UKMT website, which include some exercises for further investigation.

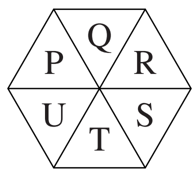
There is also a version of this document available on the UKMT website which includes each of the questions alongside its solution:

www.ukmt.org.uk

1. **D** The expression simplifies to $\dfrac{3\times 8\times 15\times 24}{(2\times 3)\times(3\times 4)\times(4\times 5)\times(5\times 6)}$. Cancelling common factors gives $\frac{1}{5}$.

2. **C** For the sum of five different primes to be prime, each of those five primes must be odd, Listing the primes starting with 3, 5, 7, 11, 13, 17, 19, . . . and working systematically through possible sums gives a smallest sum of $3+5+7+11+13=39$ which is not prime. However, the next smallest sum $3+5+7+11+17=43$ which is prime as required.

3. **C** Each of the possible parallelograms is formed from two adjacent equilateral triangles: P and Q, Q and R, R and S, S and T, T and U and finally U and P. Therefore there are six possible parallelograms.



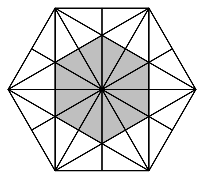
4. **D** The area of the small square is $2\times 2=4$. The area of the shaded region is then $\frac{1}{2}\times 5\times 5-\frac{1}{2}\times 2\times 2=\frac{25-4}{2}=\frac{21}{2}$. Therefore the ratio of the area of the small square to the area of the shaded region is $4:\frac{21}{2}=8:21$.

5. **C** Rewriting the calculation as $\frac{100}{101}+\frac{10}{11}+\frac{1}{1}+\frac{1}{11}+\frac{1}{101}$ shows that we can reorder the sum to give $\frac{101}{101}+\frac{11}{11}+\frac{1}{1}=3$.

6. **E** Rewriting $\dfrac{4^{800}}{8^{400}}$ using a base of 2 gives $\dfrac{(2^2)^{800}}{(2^3)^{400}}=\dfrac{2^{1600}}{2^{1200}}=2^{400}$ using rules of indices.

7. **C** Consider first the units digits of 85 and 66. Multiples of 5 can only end in 5 or 0. No multiples of 6, an even number, can end in 5. So in order that the units digit of our sum can be 0, each of the multiples of 85 and 66 must individually have units digits of 0. The smallest multiples of 85 and 66 with this property, $2\times 85=170$ and $5\times 66=330$, have sum 500. So 7 is the smallest number of stamps and they cost £5.

8. **E** By drawing extra lines from the centre of the outer hexagon to each of its vertices and from the centre to the midpoint of each edge of the outer hexagon, 12 in total, the diagram can be shown to be made of 36 congruent triangles each with angles 30°, 60° and 90°. Twelve of these triangles are shaded giving a shaded area of $\frac{1}{3} \times 216 = 72$.



9. **B** Using speed $= \dfrac{\text{distance}}{\text{time}}$ gives $3 \times 10^8 = \dfrac{d}{10^{-9}}$. Therefore $d = 3 \times 10^{-1}\,\text{m} = 0.3\,\text{m} = 30\,\text{cm}$.

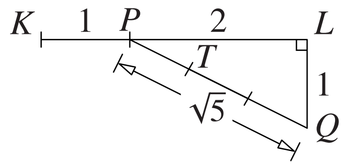
10. **D** Rearranging the equation gives $1 + 2x + 3x^2 = 9 + 6x + 3x^2$ so $1 + 2x = 9 + 6x$ and $4x = -8$. Therefore $x = -2$.

11. **A** When expressed as the product of its prime factors, $2022 = 2 \times 3 \times 337$. However, the integer n must be a factor of each integer in the middle row and so n^2 must be a factor of their product 2022. Therefore $n = 1$.

12. **E** Using the difference of two squares, the calculation we are given can be written in the form $6666666^2 - 3333333^2 = (6666666 + 3333333)(6666666 - 3333333) = 9999999 \times 3333333 = 10000000 \times 3333333 - 1 \times 3333333 = 33333330000000 - 3333333 = 33333326666667$. The sum of the digits of this integer is 63.

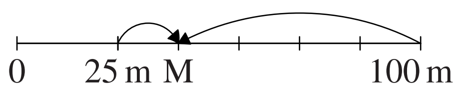
13. **C** Let the area of floor covered by exactly one rug be a, the area of floor covered by exactly two rugs be b and the area of floor covered by three rugs be c. Therefore, $a + 2b + 3c = 90$ and $a + b + c = 60$. Subtracting the second equation from the first leaves $b + 2c = 30$ and using $b = 12$ gives $c = 9$.

14. **A** Let KL be 3 units long. Then $KP = 1$, $PL = 2$ and area $KLMN = 3 \times 3 = 9$. Removing four right-angled triangles congruent to PLQ from square $KLMN$ gives area $PQRS = 9 - 4 \times \frac{1}{2} \times 1 \times 2 = 5$.The area of $PQRS$ is $\frac{5}{9}$ of the area of $KLMN$. By the same reasoning the area of $TUVW$ is $\frac{5}{9}$ of the area of $PQRS$.



Combining these proportions gives the shaded area as $\frac{5}{9} \times \frac{5}{9} = \frac{25}{81}$ of the area of $KLMN$.

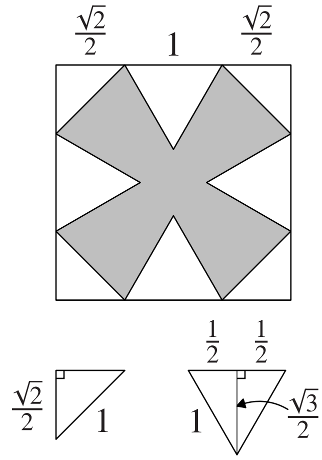
15. **B** When the hare and tortoise are moving in the same direction, the hare completes 100 m while the tortoise completes 25 m. After the hare reverses direction and the hare and tortoise are moving towards one another, the hare is still moving four times as fast.



Therefore the meeting point, M, is $\frac{4}{5}$ of 75 m = 60 m away from the finish line.

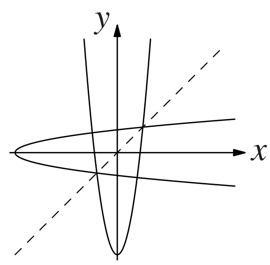
16. **B** As x and y are interchangeable in the equation, the graph must be symmetric about the line $y = x$. This excludes options C and D. Substituting $x = 0$ and $x = 1$ into the equation shows that the graph crosses the axes at $(0, 1)$ and $(1, 0)$. Note that in option E the line $x + y = 1$ meets $y = x$ at $(\frac{1}{2}, \frac{1}{2})$ whereas our curve meets $y = x$ at $(\frac{1}{4}, \frac{1}{4})$ and must therefore lie below the straight line shown in option E. The only possible option then is B.

17. B We enclose the regular octagon within a square as shown. Since the side-length of the octagon is 1, the right-angled isosceles triangles in the corners have two short sides of length $\frac{\sqrt{2}}{2}$ and so the square has side-length $1 + \sqrt{2}$. Each of the right-angled triangles has area $\frac{1}{2} \times \frac{\sqrt{2}}{2} \times \frac{\sqrt{2}}{2} = \frac{1}{4}$. Each of the equilateral triangles which were removed has base 1 and so height $\frac{\sqrt{3}}{2}$. The shaded area can be obtained as the area of the square minus that of the four isosceles corners and the four equilateral triangles; that is $(1 + \sqrt{2})^2 - 4 \times \frac{1}{4} - 4 \times \frac{1}{2} \times \frac{\sqrt{3}}{2} = 3 + 2\sqrt{2} - 1 - \sqrt{3} = 2 + 2\sqrt{2} - \sqrt{3}$.

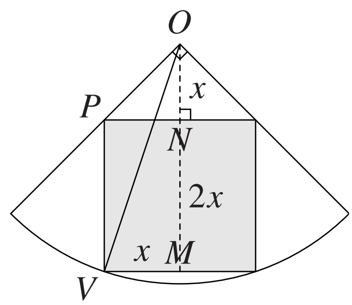


18. B Let $3^x = X$ and $3^y = Y$. The two equations can then be written as $X + 3Y = 5\sqrt{3}$ and $3X + Y = 3\sqrt{3}$. Subtracting three lots of the second equation from the first gives $-8X = -4\sqrt{3}$ so $X = \frac{\sqrt{3}}{2}$. Subtracting three lots of the first equation from the second gives $-8Y = -12\sqrt{3}$ so $Y = \frac{3\sqrt{3}}{2}$. The value of $3^x + 3^y = X + Y = \frac{\sqrt{3}}{2} + \frac{3\sqrt{3}}{2} = 2\sqrt{3}$. Alternatively, we could add the two equations giving $4X + 4Y = 8\sqrt{3}$. Dividing by 4, $X + Y = 3^x + 3^y = 2\sqrt{3}$ without knowing the value of either 3^x or 3^y individually.

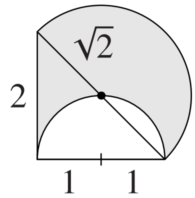
19. E The first equation can be rearranged to the form $y = x^2 - 2022$ which is a translation of $y = x^2$ down 2022 units. The second equation is a reflection of the first, in the line $y = x$. There are four points of intersection of these two parabolas.



20. D Let O be the centre of the circle, M and N be the midpoints of two sides of the square and V and P be vertices of two sides of the square as shown. Line ONM is a line of symmetry. Let $ON = x$. Therefore $NP = x$ as ON and NP are sides of the right-angled isosceles triangle ONP. Also, $PV = MN = 2x$. Consider right-angled triangle OVM. The radius of the circle is given as 10, therefore $(3x)^2 + x^2 = 10^2$ so $10x^2 = 100$ and $x^2 = 10$. Hence the area of the square is $(2x)^2 = 4x^2 = 4 \times 10 = 40$.

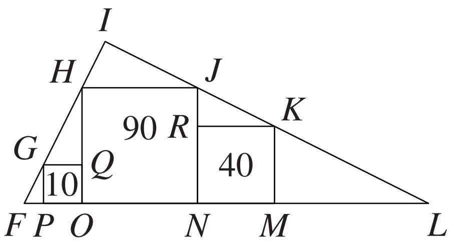


21. D Half the diagram is shown here. In it, the shaded area equals the area of a right-angled isosceles triangle of side-length 2 plus the area of a large semicircle minus the area of a small semicircle of radius 1. Using Pythagoras' Theorem, the diameter of the large semicircle has length $2\sqrt{2}$ and so the radius is $\sqrt{2}$. Therefore the shaded area of the full diagram is $2[\frac{1}{2} \times 2 \times 2 + \frac{1}{2}\pi \times (\sqrt{2})^2 - \frac{1}{2}\pi \times 1^2] = 2(2 + \pi - \frac{1}{2}\pi) = 4 + \pi$.



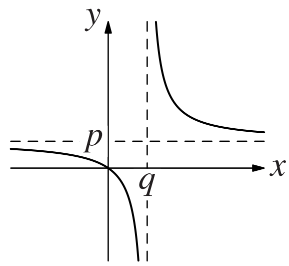
22. **B** Squaring both sides of the equation gives $x - \sqrt{x+23} = 8 - 4\sqrt{2}y + y^2$ which can be rearranged to $\sqrt{x+23} = (x - 8 - y^2) + 4\sqrt{2}y$ [1]. Squaring equation [1] gives $x + 23 = (x - 8 - y^2)^2 + 2(4\sqrt{2}y)(x - 8 - y^2) + 32y^2$ [2]. We are given that both x and y are integers and so the surd component, $2(4\sqrt{2}y)(x - 8 - y^2)$, must equal 0. Therefore either $y = 0$ or $(x - 8 - y^2) = 0$ [3]. Consider first the case $y = 0$. Here, equation [2] reduces to $x + 23 = (x - 8)^2$. This expands to $x^2 - 17x + 41 = 0$ which has no integer solutions as its discriminant is $(-17)^2 - 4 \times 1 \times 41 = 125$, which is not square. Secondly considering $(x - 8 - y^2) = 0$ [3] reduces [1] to $\sqrt{x+23} = 4\sqrt{2}y$ and therefore $x + 23 = 32y^2$. Using [3] again gives $x = 8 + y^2$ and so $31 + y^2 = 32y^2$. Therefore $y^2 = 1$. Hence $y = \pm 1$ and in either case, $x = 8 + 1 = 9$. Because equations have been squared, some solutions could be spurious. Substituting in the original equation, we see that $(9, 1)$ is a solution but $(9, -1)$ is not. Hence there is just one solution.

23. **A** The lengths of the sides of the three squares are $\sqrt{10}$, $3\sqrt{10}$ and $2\sqrt{10}$ respectively. Therefore $HQ = 2\sqrt{10}$ and $RJ = \sqrt{10}$. In triangle GQH, the gradient of GH is $\frac{2\sqrt{10}}{\sqrt{10}} = 2$. In triangle JRK, the gradient of JK is $\frac{-\sqrt{10}}{2\sqrt{10}} = \frac{-1}{2}$. Therefore lines FI (on which GH lies) and IL (on which JK lies) are perpendicular.

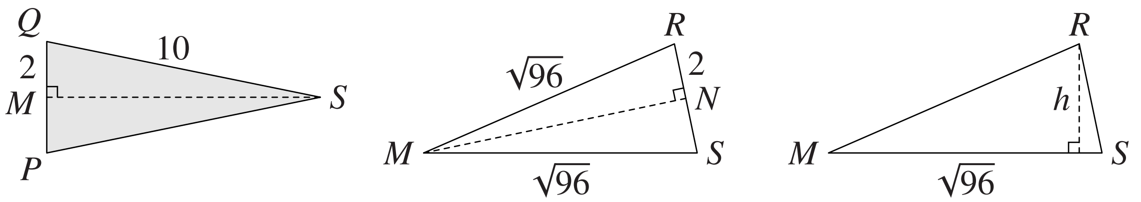


All five right-angled triangles around the edge of the figure and triangle FIL itself are similar as they contain the same angles. They all have sides in the ratio $1 : 2 : \sqrt{5}$. To calculate the area of triangle FIL we need the length IL, as the area of $FIL = \frac{1}{2} \times IL \times \frac{1}{2}IL$. The length IL is made of three sections: $JK = \sqrt{10} \times \sqrt{5}$, $KL = 2JK = 2 \times \sqrt{10} \times \sqrt{5}$ and $IJ = \frac{2}{\sqrt{5}} \times HJ = \frac{2}{\sqrt{5}} \times 3\sqrt{10}$. Therefore $IL = IJ + JK + KL = 6\sqrt{2} + \sqrt{50} + 2\sqrt{50} = 21\sqrt{2}$. Hence the area of triangle $FIL = \frac{1}{2} \times 21\sqrt{2} \times \frac{21}{2}\sqrt{2} = 220.5$.

24. **D** Rearranging $xy = px + qy$ to make y the subject, gives $xy - qy = px$ so $y(x - q) = px$ and therefore $y = \frac{px}{x-q}$ which rearranges to $y = p + \frac{pq}{x-q}$. A sketch of the graph of this function for real values of x and y is shown. As x and y are both integers in this question, y takes its maximum value when $x - q$ is as small as possible therefore $x - q = 1$ so $x = q + 1$. The expression $y - x$ then becomes $\dfrac{px}{1} - x = (p - 1)x = (p - 1)(q + 1)$.



25. **A**



Let M be the midpoint of QP. The volume of the carton is $\frac{1}{3} \times$ base area of triangle $PQS \times$ the perpendicular height from R to the plane containing PQS. Triangle PQS is isosceles and $MS = \sqrt{10^2 - 2^2} = \sqrt{96}$. So area of $PQS = \frac{1}{2} \times 4 \times \sqrt{96} = 8\sqrt{6}$. Consider isosceles triangle MRS and let N be the midpoint of RS. $MN = \sqrt{(\sqrt{96})^2 - 2^2} = \sqrt{92}$, so with RS as the base, area of $MRS = \frac{1}{2} \times 4 \times \sqrt{92} = 4\sqrt{23}$. Now with MS as the base, area of $MRS = \frac{1}{2} \times MS \times h$. Therefore $4\sqrt{23} = \frac{1}{2} \times \sqrt{96} \times h$ and $h = \frac{2\sqrt{23}}{\sqrt{6}}$. Finally, the volume $= \frac{1}{3} \times 8\sqrt{6} \times \frac{2\sqrt{23}}{\sqrt{6}} = \frac{16\sqrt{23}}{3}$.